Exercise 3.3.1

For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

(a) $f(x) = 1$

(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

Solution

Assume that $f(x)$ is a piecewise smooth function on the interval $0 \leq x \leq L$. The even extension of this function to the whole line $(-\infty < x < \infty)$ with period $2L$ is given by the Fourier cosine series expansion,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \tag{1}$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The odd extension of $f(x)$ to the whole line with period $2L$ is given by the Fourier sine series expansion,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \tag{2}$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. To determine A_0, integrate both sides of equation (1) with respect to x from 0 to L.

$$\int_0^L f(x)\,dx = \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}\right) dx$$

Split up the integral on the right and bring the constants in front.

$$\begin{aligned}\int_0^L f(x)\,dx &= A_0 \underbrace{\int_0^L dx}_{=L} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L}\,dx}_{=0} \\ &= A_0 L\end{aligned}$$

Therefore,

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx.}$$

To get A_n, multiply both sides of equation (1) by $\cos \frac{p\pi x}{L}$

$$f(x) \cos \frac{p\pi x}{L} = A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L f(x)\cos\frac{p\pi x}{L}\,dx = \int_0^L \left(A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)dx$$

Split up the integral on the right and bring the constants in front.

$$\int_0^L f(x)\cos\frac{p\pi x}{L}\,dx = A_0\underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx$$

Because the cosine functions are orthogonal, the integral on the right is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$\begin{aligned}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx &= A_n \int_0^L \cos^2\frac{n\pi x}{L}\,dx \\ &= A_n\left(\frac{L}{2}\right)\end{aligned}$$

Therefore,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.}$$

To get B_n, multiply both sides of equation (2) by $\sin\frac{p\pi x}{L}$

$$f(x)\sin\frac{p\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L f(x)\sin\frac{p\pi x}{L}\,dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx$$

Split up the integral on the right and bring the constants in front.

$$\int_0^L f(x)\sin\frac{p\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the right is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$\begin{aligned}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx &= B_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx \\ &= B_n\left(\frac{L}{2}\right)\end{aligned}$$

Therefore,

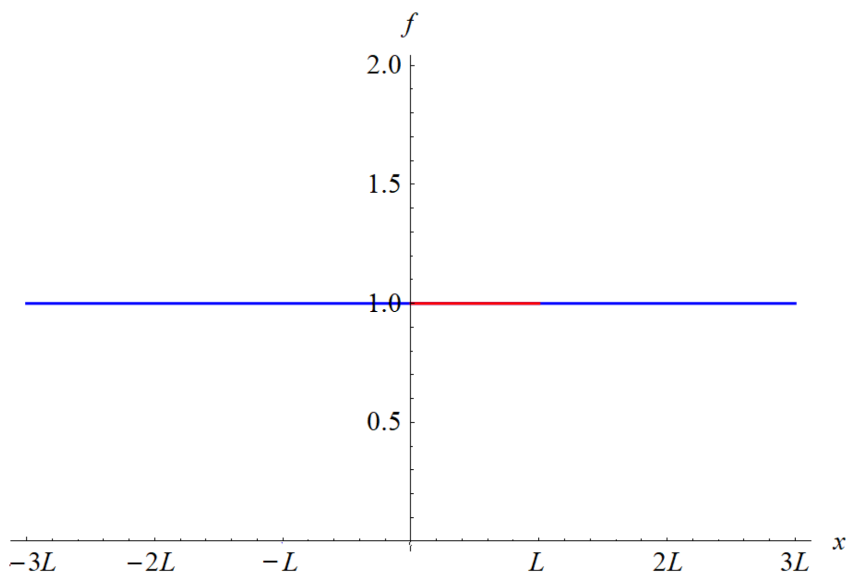
$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Part (a)

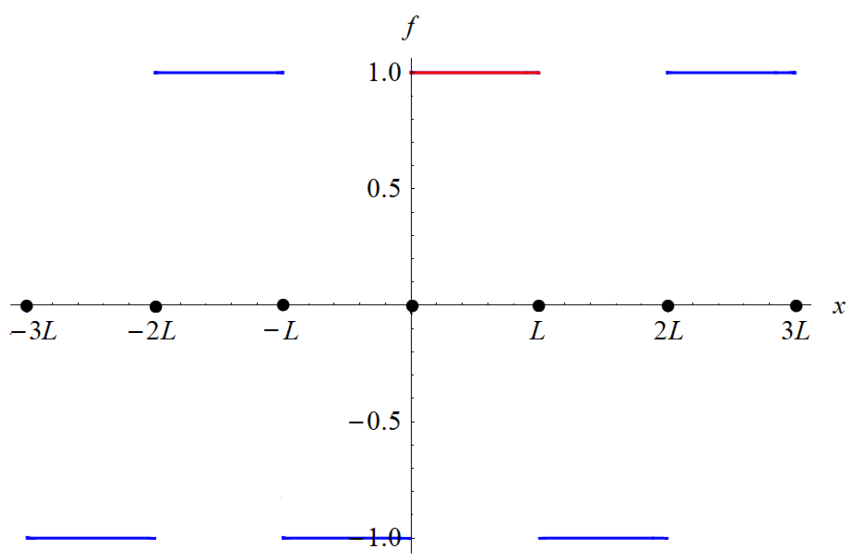
For $f(x) = 1$, the coefficients are

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L dx = 1 \\ A_n &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\,dx = 0 \\ B_n &= \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{2[1-(-1)^n]}{n\pi}. \end{aligned}$$

The even extension of $f(x)$ to the whole line is shown below.



The odd extension of $f(x)$ to the whole line is shown below.

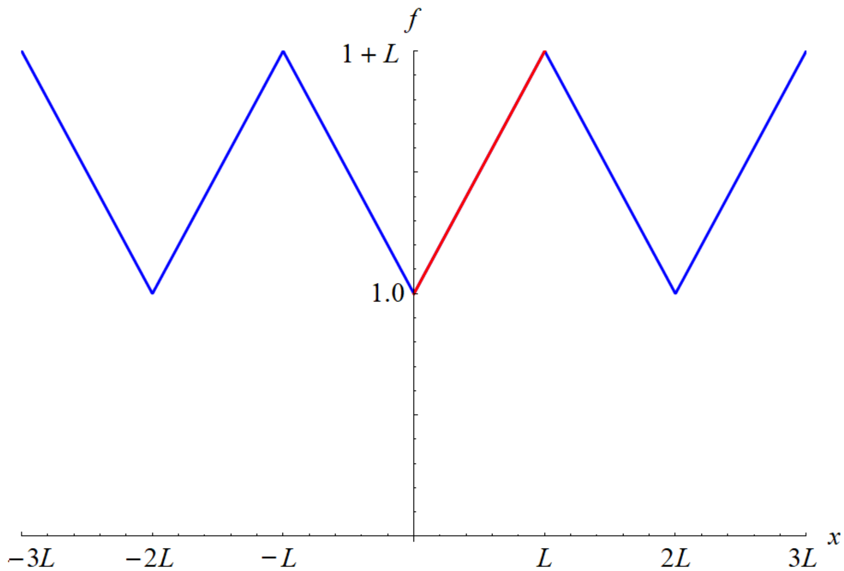


Part (b)

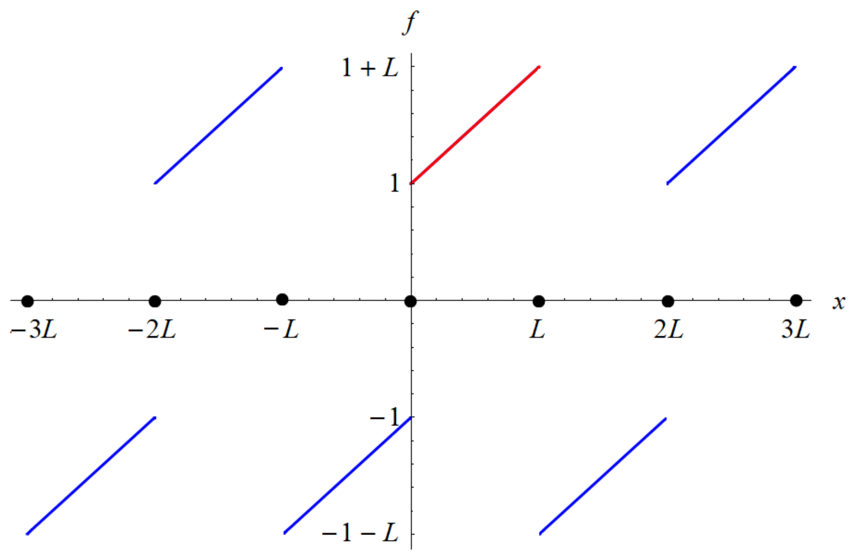
For $f(x) = 1 + x$, the coefficients are

$$A_0 = \frac{1}{L}\int_0^L (1+x)\,dx = \frac{L+2}{2}$$
$$A_n = \frac{2}{L}\int_0^L (1+x)\cos\frac{n\pi x}{L}\,dx = \frac{2[-1+(-1)^n]L}{n^2\pi^2}$$
$$B_n = \frac{2}{L}\int_0^L (1+x)\sin\frac{n\pi x}{L}\,dx = \frac{2-2(-1)^n(L+1)}{n\pi}.$$

The even extension of $f(x)$ to the whole line is shown below.



The odd extension of $f(x)$ to the whole line is shown below.

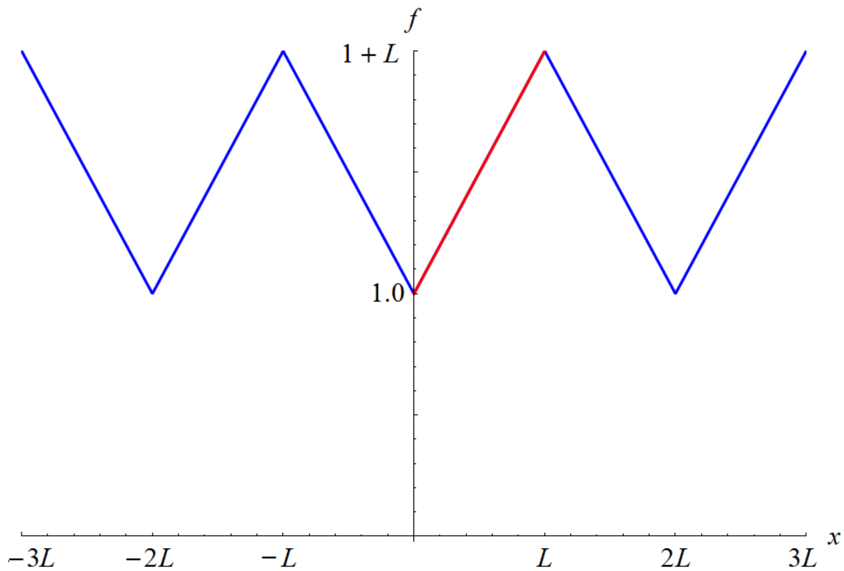


Part (c)

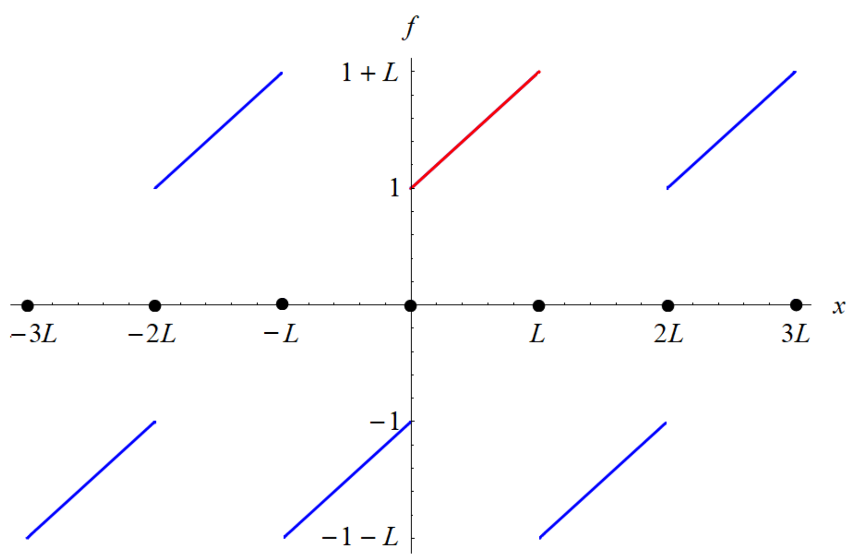
For $f(x) = x$ if $x < 0$ and $f(x) = 1 + x$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{L}\int_0^L (1+x)\,dx = \frac{L+2}{2}$$
$$A_n = \frac{2}{L}\int_0^L (1+x)\cos\frac{n\pi x}{L}\,dx = \frac{2[-1+(-1)^n]L}{n^2\pi^2}$$
$$B_n = \frac{2}{L}\int_0^L (1+x)\sin\frac{n\pi x}{L}\,dx = \frac{2-2(-1)^n(L+1)}{n\pi}.$$

The even extension of $f(x)$ to the whole line is shown below.



The odd extension of $f(x)$ to the whole line is shown below.

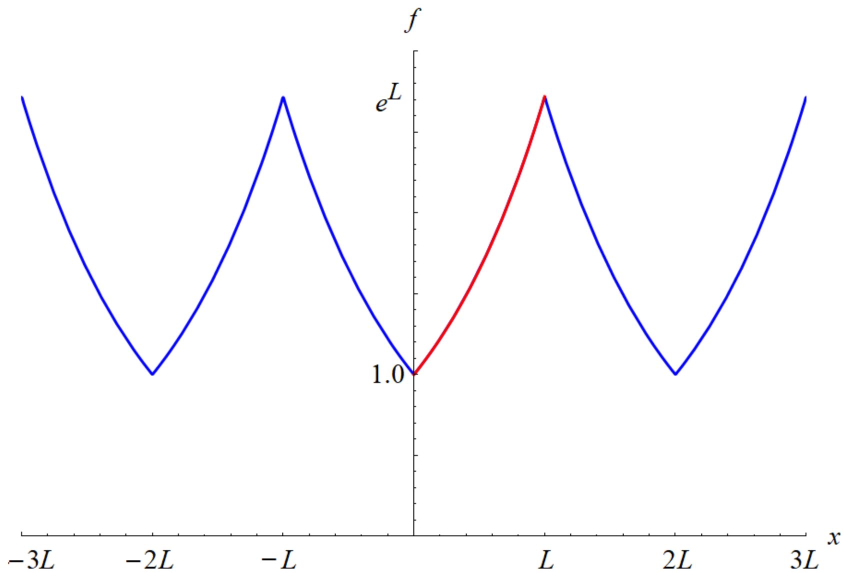


Part (d)

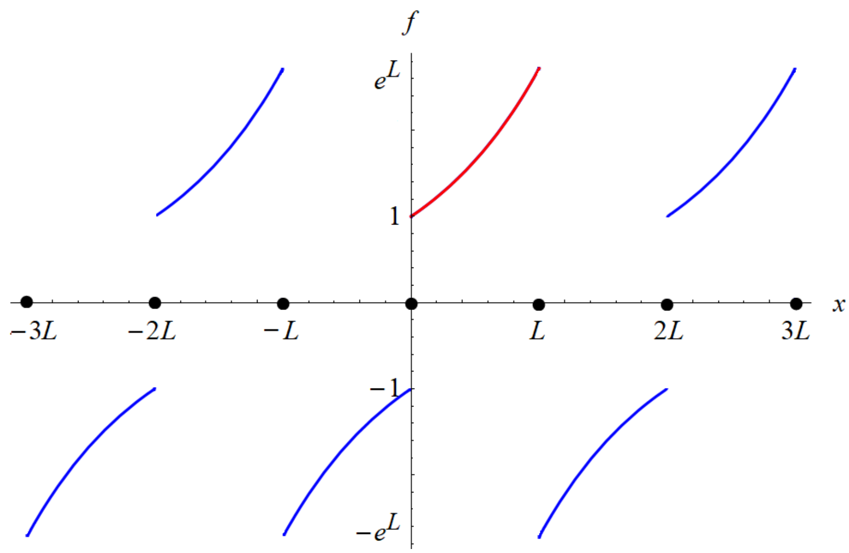
For $f(x) = e^x$, the coefficients are

$$A_0 = \frac{1}{L}\int_0^L e^x\,dx = \frac{e^L - 1}{L}$$
$$A_n = \frac{2}{L}\int_0^L e^x \cos\frac{n\pi x}{L}\,dx = \frac{2[-1 + (-1)^n e^L]L}{n^2\pi^2 + L^2}$$
$$B_n = \frac{2}{L}\int_0^L e^x \sin\frac{n\pi x}{L}\,dx = -\frac{2[-1 + (-1)^n e^L]n\pi}{n^2\pi^2 + L^2}.$$

The even extension of $f(x)$ to the whole line is shown below.



The odd extension of $f(x)$ to the whole line is shown below.

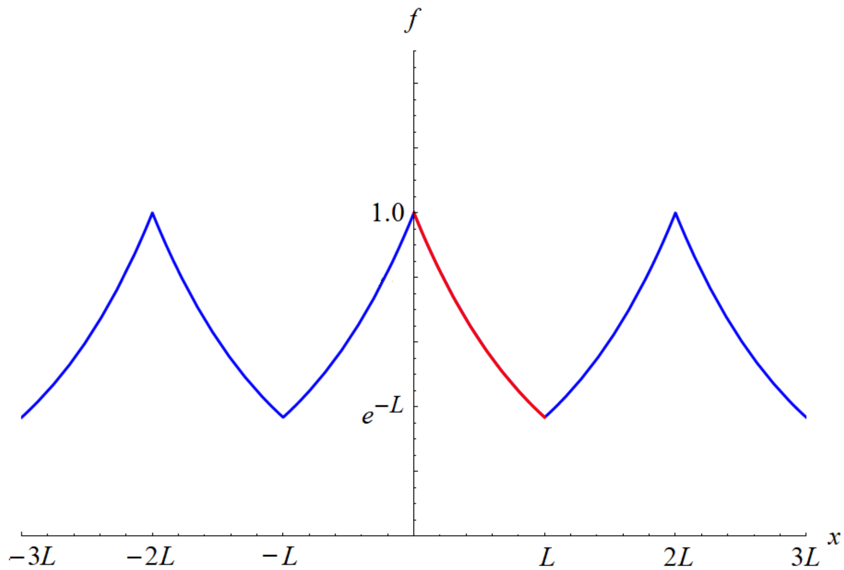


Part (e)

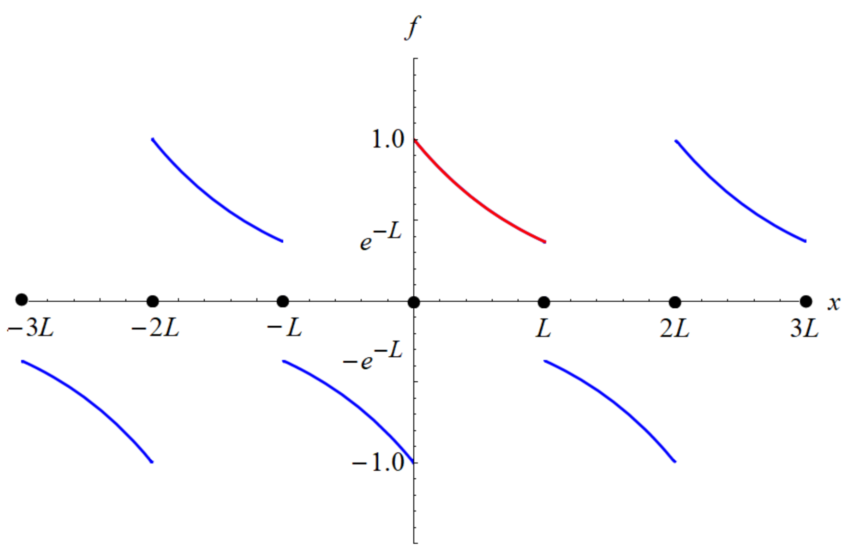
For $f(x) = 2$ if $x < 0$ and $f(x) = e^{-x}$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{L}\int_0^L e^{-x}\,dx = \frac{1-e^{-L}}{L}$$
$$A_n = \frac{2}{L}\int_0^L e^{-x}\cos\frac{n\pi x}{L}\,dx = \frac{2[1-(-1)^n e^{-L}]L}{n^2\pi^2+L^2}$$
$$B_n = \frac{2}{L}\int_0^L e^{-x}\sin\frac{n\pi x}{L}\,dx = \frac{2[1-(-1)^n e^{-L}]n\pi}{n^2\pi^2+L^2}.$$

The even extension of $f(x)$ to the whole line is shown below.



The odd extension of $f(x)$ to the whole line is shown below.



Now assume that $f(x)$ is a piecewise smooth function on the interval $-L \leq x \leq L$. The $2L$-periodic extension of this function to the whole line ($-\infty < x < \infty$) is given by the Fourier series expansion,

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right), \tag{3}$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. To get A_0, integrate both sides of equation (3) with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L}\left[A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)\right]dx \\ &= A_0 \underbrace{\int_{-L}^{L} dx}_{=\,2L} + \sum_{n=1}^{\infty}\left(A_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx}_{=\,0} + B_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx}_{=\,0}\right) \\ &= A_0(2L)\end{aligned}$$

Therefore,

$$\boxed{A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.}$$

To get A_n, multiply both sides of equation (3) by $\cos\frac{p\pi x}{L}$

$$f(x)\cos\frac{p\pi x}{L} = A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + B_n \sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left[A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + B_n \sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)\right]dx \\ &= A_0 \underbrace{\int_{-L}^{L}\cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty}\left(A_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx + B_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx}_{=\,0}\right)\end{aligned}$$

Because the sine and cosine functions are orthogonal, the third integral on the right is zero. The second integral is also zero if $n \neq p$. Only if $n = p$ does this integral yield a nonzero result.

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx &= A_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx \\ &= A_n(L)\end{aligned}$$

Therefore,

$$\boxed{A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx.}$$

To get B_n, multiply both sides of equation (3) by $\sin \frac{p\pi x}{L}$

$$f(x)\sin\frac{p\pi x}{L} = A_0 \sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left[A_0 \sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)\right]dx \\ &= A_0 \underbrace{\int_{-L}^{L}\sin\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty}\left(A_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx}_{=\,0} + B_n \int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx\right)\end{aligned}$$

Because the sine and cosine functions are orthogonal, the second integral on the right is zero. The third integral is also zero if $n \neq p$. Only if $n = p$ does this integral yield a nonzero result.

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx &= B_n \int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx \\ &= B_n(L)\end{aligned}$$

Therefore,

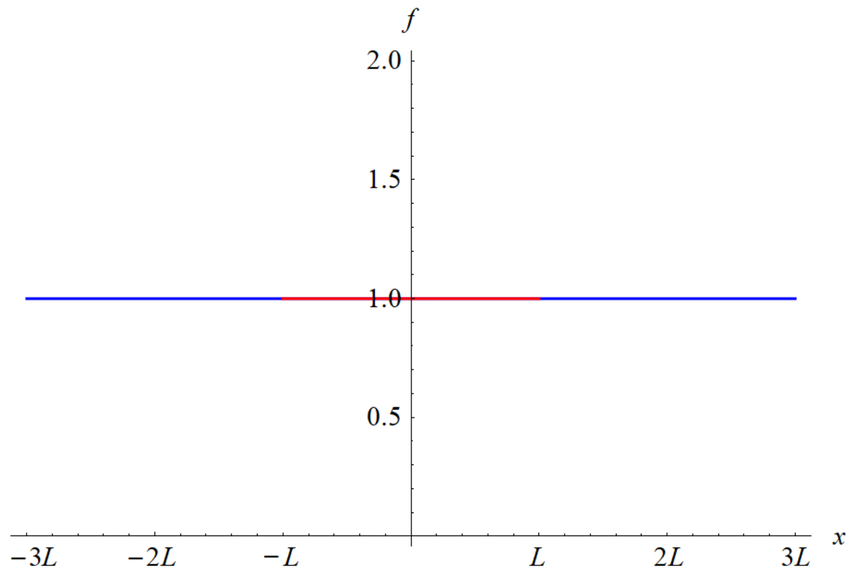
$$\boxed{B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Part (a)

For $f(x) = 1$, the coefficients are

$$\begin{aligned}A_0 &= \frac{1}{2L}\int_{-L}^{L} dx = 1 \\ A_n &= \frac{1}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = 0 \\ B_n &= \frac{1}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx = 0.\end{aligned}$$

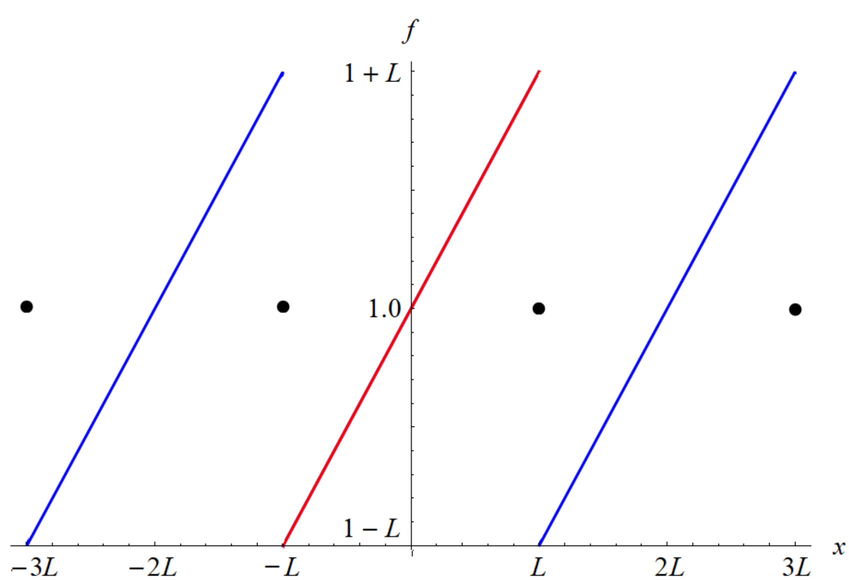
The $2L$-periodic extension of $f(x)$ to the whole line is shown below.



Part (b)

For $f(x) = 1 + x$, the coefficients are

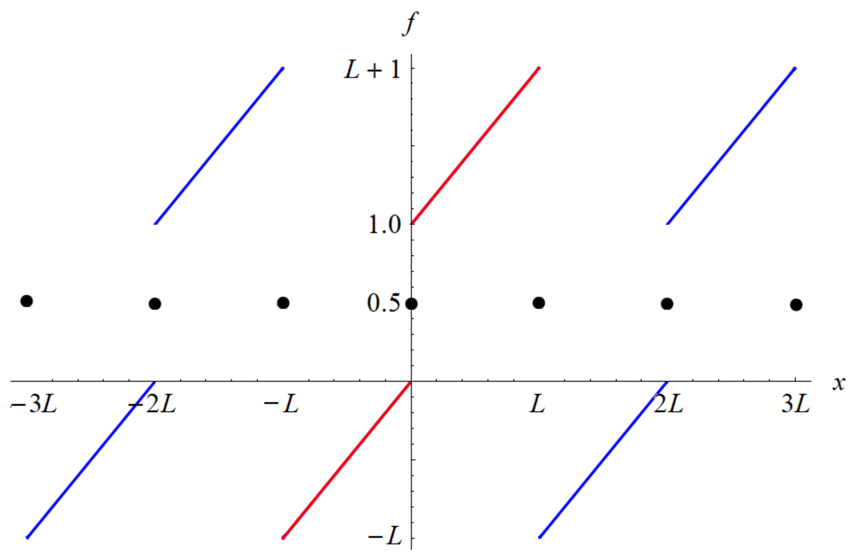
$$A_0 = \frac{1}{2L}\int_{-L}^{L}(1+x)\,dx = 1$$
$$A_n = \frac{1}{L}\int_{-L}^{L}(1+x)\cos\frac{n\pi x}{L}\,dx = 0$$
$$B_n = \frac{1}{L}\int_{-L}^{L}(1+x)\sin\frac{n\pi x}{L}\,dx = -\frac{2(-1)^n L}{n\pi}.$$



Part (c)

For $f(x) = x$ if $x < 0$ and $f(x) = 1 + x$ if $x > 0$, the coefficients are

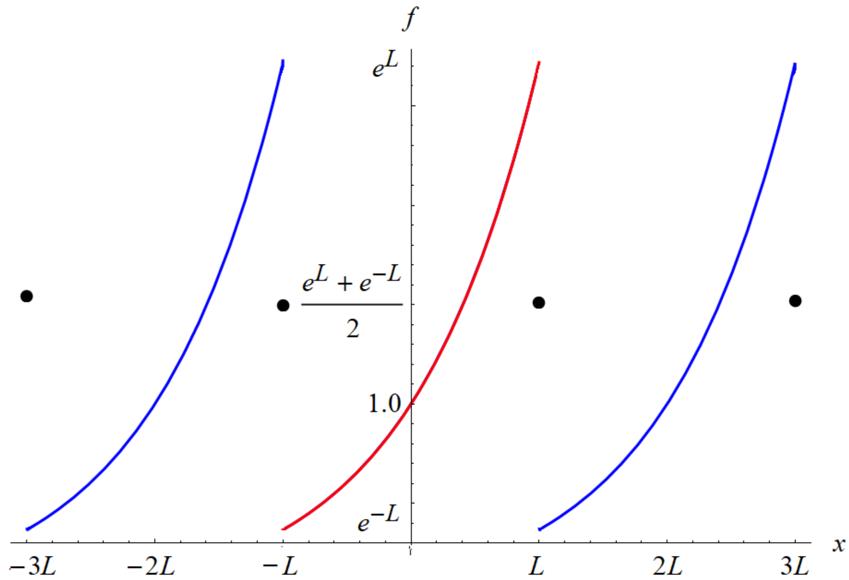
$$A_0 = \frac{1}{2L}\left[\int_{-L}^{0} x\,dx + \int_0^L (1+x)\,dx\right] = \frac{1}{2}$$
$$A_n = \frac{1}{L}\left[\int_{-L}^{0} x\cos\frac{n\pi x}{L}\,dx + \int_0^L (1+x)\cos\frac{n\pi x}{L}\,dx\right] = 0$$
$$B_n = \frac{1}{L}\left[\int_{-L}^{0} x\sin\frac{n\pi x}{L}\,dx + \int_0^L (1+x)\sin\frac{n\pi x}{L}\,dx\right] = \frac{1-(-1)^n(1+2L)}{n\pi}.$$



Part (d)

For $f(x) = e^x$, the coefficients are

$$A_0 = \frac{1}{2L}\int_{-L}^{L} e^x\,dx = \frac{\sinh L}{L}$$
$$A_n = \frac{1}{L}\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\,dx = \frac{2(-1)^n L\sinh L}{n^2\pi^2 + L^2}$$
$$B_n = \frac{1}{L}\int_{-L}^{L} e^x\sin\frac{n\pi x}{L}\,dx = -\frac{2(-1)^n n\pi\sinh L}{n^2\pi^2 + L^2}.$$



Part (e)

For $f(x) = 2$ if $x < 0$ and $f(x) = e^{-x}$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} 2\,dx + \int_0^L e^{-x}\,dx\right) = 1 + \frac{1-e^{-L}}{2L}$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{0} 2\cos\frac{n\pi x}{L}\,dx + \int_0^L e^{-x}\cos\frac{n\pi x}{L}\,dx\right) = \frac{[1-(-1)^n e^{-L}]L}{n^2\pi^2 + L^2}$$

$$B_n = \frac{1}{L}\left(\int_{-L}^{0} 2\sin\frac{n\pi x}{L}\,dx + \int_0^L e^{-x}\sin\frac{n\pi x}{L}\,dx\right) = \frac{[1-(-1)^n e^{-L}]n\pi}{n^2\pi^2 + L^2} + \frac{2[-1+(-1)^n]}{n\pi}.$$